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The dynamical symmetry of isotropic systems

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Abstract. In terms of Noether's theorem, we propose a systematic approach to attack the dynamical symmetry problem. Starting with a polynomial symmetry transformation, we can simultaneously determine the transformation functions and the possible potentials, then obtain the corresponding conserved quantities and the symmetry algebras. In the isotropic case, up to a certain order of transformations, we find that all possible systems which possess symmetry larger than a superficial geometrical symmetry fall into five categories: free particle, harmonic oscillator, Coulomb potential, centripetal potential and mixed potential. The symmetry algebras of these systems are also discussed.

1. Introduction

Dynamical symmetry has been discussed for a long time in both classical mechanics and quantum mechanics. (For a review, see Mariwalla (1975), McIntosh (1971) and references therein.) This concept is particularly useful in quantum mechanics because it determines the spacing and the degeneracy of the energy levels, and facilitates the calculation of matrix elements and the derivation of the selection rule (Wybourne 1974, Englefield 1972). However, there is a systematic way to find such symmetries, so traditionally they have been called 'accidental' or 'hidden' symmetries. Therefore, it is useful to develop a general procedure for investigating such dynamical symmetry for an arbitrary physical system (Mariwalla 1975). In addition, there is a hope of finding a systematic way by which one can determine all possible systems which possess dynamical symmetry. It is preferable, of course, to solve both aspects of this problem at the same time, if possible.

A usual method of searching for dynamical symmetry is as follows. Firstly, one tries to find the constants of the motion of a classical system. Then one calculates the Poisson bracket; then, after selecting the proper ordering of the canonical variables as they enter in the constants of the motion, one transcribes all expressions into quantum mechanics using the quantum bracket instead of the Poisson bracket. Therefore, the main task in searching for dynamical symmetry is to discover the constants of the motion.

In dealing with a symmetry and the associated conserved quantity, Noether's theorem (Noether 1918, Gel'fand and Fomin 1963) is a very powerful tool. Although some new methods have been developed recently (e.g. Katzin and Levine 1968, 1974, Wollenberg 1975), in this paper we apply Noether's theorem to give a systematic way in which one can determine the possible systems (potentials) and the corresponding dynamical symmetries simultaneously. In § 2, we give a generalised form of Noether's

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theorem. In terms of this theorem we can determine the polynomial symmetry transformation and the corresponding constants of the motion. In § 3 we use this generalised Noether theorem to attack the dynamical symmetry problem of the three-dimensional isotropic single-particle system. We find that, up to a certain order of polynomial transformation, the possible potentials which possess a symmetry larger than the superficial geometrical symmetry are the free particle, the harmonic oscillator, the Coulomb potential, the centripetal potential and the mixed potential. Then in § 4 we give the explicit constants of the motion corresponding to various possible systems and discuss briefly the symmetry algebra. Finally, we make a few concluding remarks on our results in § 5.

2. Noether's theorem

Noether's theorem has many forms. In 1978, Lutzky presented an expression which is very convenient for our purpose. He states Noether's theorem as follows.

If the equation of motion of a physical system described by a Lagrangian $\mathscr{L}(q, \dot{q}, t)$ is invariant under the transformation

$$Q = e^{\lambda G(q,t)} q, \qquad T = e^{\lambda G} t, \qquad \dot{Q} = e^{\lambda E(q,\dot{q},t)} \dot{q}, \tag{1}$$

namely, if the Lagrangian satisfies the equation

$$E\{\mathscr{L}\} = -\dot{\xi}\mathscr{L} + \dot{f}, \tag{2}$$

then we obtain a constant of the motion Φ,

$$\Phi = (\xi\dot{q} - \eta)\partial\mathscr{L}/\partial\dot{q} - \xi\mathscr{L} + f. \tag{3}$$

Here λ is a group parameter, ξ, η and f are functions of time and the coordinate. G and E are differential operators:

$$G = \xi(q, t)\frac{\partial}{\partial t} + \eta(q, t)\frac{\partial}{\partial q}, \qquad E = G + (\dot{\eta} - \dot{q}\dot{\xi})\frac{\partial}{\partial \dot{q}}. \tag{4}$$

Through direct calculation, one can verify that the total time derivative of Φ is zero by means of the Euler–Lagrange equation. Besides providing the constant of the motion, this expression of the theorem also gives an equation which the symmetry transformation and the Lagrangian should satisfy. So it allows us to find the symmetry transformation for a given system, or, vice versa, to determine the potential for a known symmetry transformation. Fortunately, it can even be used to fix both the symmetry and potential at the same time. However, this presentation of the theorem only deals with the point transformation; the constant of motion cannot contain a term with momentum of power greater than 2. So we should extend the transformation to include higher time derivatives of the coordinates (Lévy-Leblond 1971). Anderson *et al* (1974) have done something analogous for generalising Lie's counting theorem. Of course, since the equation of motion (which is a second-order differential equation) is imposed, it is sufficient that the transformations contain the first-order derivative of coordinates only. In addition, when the coordinate transformations contain derivatives, the time transformation need not be taken into account because the corresponding constants of the motion can be obtained by appropriately choosing the coordinate transformations.

Now we state Noether's theorem in three-dimensional space.

If the equation of motion of a multiparticle system described by Lagrangian $\mathscr{L}(r_i^a, \dot{r}_i^a, t)$ is invariant under the infinitesimal transformation

$$\delta r_i^a = \eta_i^a(r_i^b, \dot{r}_i^b, t), \tag{5}$$

i.e. the Lagrangian satisfies the equation

$$\eta_i^a \partial\mathscr{L}/\partial r_i^a + \dot{\eta}_i^a \partial\mathscr{L}/\partial \dot{r}_i^a = -\dot{f}, \tag{6}$$

then a constant of the motion is given by

$$\Phi = \eta_i^a \partial\mathscr{L}/\partial \dot{r}_i^a + f. \tag{7}$$

Here f is an arbitrary function of time and coordinates only.

Applying this theorem to a single-particle isotropic system, we have the following equations:

$$\mathscr{L} = \tfrac{1}{2}\dot{r}_i^2 - V(r), \tag{8}$$

$$\delta r_i = \eta_i(r_j, \dot{r}_j, t), \tag{9}$$

$$-\eta_i \partial V/\partial r_i + \dot{\eta}_i \dot{r}_i + \dot{f} = 0, \tag{10}$$

$$\Phi = \eta_i \dot{r}_i + f = \eta_i p_i + f. \tag{11}$$

3. Potentials and symmetry transformations

To solve equation (10) we need to impose a restriction on the transformation function η_i. Generally, we can expand the η_i in a power series of velocity $\dot{r}_i$ (momentum) and the coordinates

$$\eta_i = \sum_{\substack{\alpha_1\alpha_2\alpha_3 \\ \beta_1\beta_2\beta_3}} a_{i\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}(t) r_1^{\alpha_1} r_2^{\alpha_2} r_3^{\alpha_3} \dot{r}_1^{\beta_1} \dot{r}_2^{\beta_2} \dot{r}_3^{\beta_3}. \tag{12}$$

Corresponding to this polynomial symmetry transformation, we obtain a polynomial constant of the motion

$$\Phi = \sum_{i=1}^{3} \sum_{\alpha,\beta} a_{i\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} r_1^{\alpha_1} r_2^{\alpha_2} r_3^{\alpha_3} \dot{r}_1^{\beta_1} \dot{r}_2^{\beta_2} \dot{r}_3^{\beta_3} \dot{r}_i + f. \tag{13}$$

This provides a systematic way to search for algebraic constants of the motion. Expansion (12) can be slightly simplified by considering the vector property of η_i.

As an example, we truncate the expansion to a certain order; we also suppose that the leading term of η_i is a bilinear function of velocities and coordinates. Then we can express η_i in the following way:

$$\eta_i = \theta_k r_k \dot{r}_i + \rho_k \dot{r}_k r_i + r_k \dot{r}_k \sigma_i + \xi_{ij} \dot{r}_j + \zeta_i. \tag{14}$$

Here, θ_k, ρ_k, σ_k and the symmetric tensor ξ_{ij} (the antisymmetric tensor does not lead to the constant of the motion) are functions of time only, and ζ_i is a function of time and coordinates.

From equations (14) and (10), using the equation of motion

$$\ddot{r}_i = -(r_i/r)V' \tag{15}$$

and considering that $\dot{r}_i$ is an independent variable, we obtain the following equations:

$$(\theta_k + \rho_k + \sigma_k)\dot{r}_k\dot{r}^2 = 0, \tag{16}$$

$$\dot{\theta}_k r_k \dot{r}^2 + \dot{\rho}_k \dot{r}_k r_i \dot{r}_i + \dot{\sigma}_i \dot{r}_i r_k \dot{r}_k + \dot{\xi}_{ij}\dot{r}_i\dot{r}_j + \zeta_{i,j}\dot{r}_i\dot{r}_j = 0, \tag{17}$$

$$-(2\theta_k + \sigma_k + \rho_k)r_k r_i \dot{r}_i V'/r - (\sigma_i + \rho_i)rV'\dot{r}_i - (\xi_{ij} + \xi_{ji})r_j\dot{r}_i V'/r + \zeta_{i,0}\dot{r}_i + f_{,i}\dot{r}_i = 0, \tag{18}$$

$$-\zeta_i r_i V'/r + f_{,0} = 0. \tag{19}$$

Here we have used the notations

$$f_{,0} = \frac{\partial f}{\partial t}, \qquad f_{,i} = \frac{\partial f}{\partial r_i}, \qquad V' = \frac{\mathrm{d}V}{\mathrm{d}r}. \tag{20}$$

From equation (16) we have

$$\theta_k + \rho_k + \sigma_k = 0. \tag{21}$$

Then putting (21) back into equation (17), we obtain

$$\zeta_{i,j} = \tfrac{1}{2}(\dot{\theta}_i r_j + \dot{\theta}_j r_i) - \dot{\theta}_k r_k \delta_{ij} - \dot{\xi}_{ij} + D_l \varepsilon_{lij}. \tag{22}$$

Symmetry under interchange of the order of differentiation imposes

$$\zeta_{i,jk} = \zeta_{i,kj} \tag{23}$$

which results in

$$\dot{\theta}_k = 0, \qquad D_{l,k} = 0. \tag{24}$$

Then we have

$$\zeta_i = -\dot{\xi}_{ij} r_i + D_l \varepsilon_{lij} r_j + \zeta_i^0(t). \tag{25}$$

Feeding (21) and (25) back into equations (18) and (19), and using the condition

$$f_{,ij} = f_{,ji}, \tag{26}$$

we produce some constraints on D_l, θ_i, and ξ_{ij}:

$$\text{(a)} \quad D_{l,0} = 0, \tag{27}$$

$$\text{(b)} \quad \begin{cases} \theta_i \neq 0, & V'' + (2/r)V' = 0, \\ \theta_i = 0, & \text{otherwise.} \end{cases} \tag{28}$$

This means that only for the Coulomb potential

$$V = a_{-1}/r \tag{29}$$

or the free particle case could θ_i be non-zero. We divide the tensor ξ_{ij} into its trace part and its traceless part

$$\text{(c)} \quad \xi_{ij} = \xi\delta_{ij} + \xi_{ij}^0, \qquad \xi_{ii}^0 = 0. \tag{30}$$

Then condition (26) also yields

$$\begin{cases} \xi_{ij}^0 \neq 0, & V' - rV'' = 0, \\ \xi_{ij}^0 = 0, & \text{otherwise,} \end{cases} \tag{31}$$

i.e. only when the potential is the harmonic oscillator

$$V=a_2r^2 \tag{32}$$

or free particle is the traceless part of ξ_{ij} non-vanishing.

In parallel with condition (26), there is another condition

$$f_{,i0}=f_{,0i} \tag{33}$$

which results in an analogous equation:

$$\dddot{\xi}r_i+\dot{\xi}r_i\left(\frac{3}{r}V'+V''\right)+\dddot{\xi}^0_{ij}r_j+4\dot{\xi}^0_{ij}\frac{r_j}{r}V'+\frac{\dot{\xi}^0_{lm}}{r^3}r_lr_mr_i(-V'+rV'')$$

$$-\ddot{\zeta}^0_i-\zeta^0_i\frac{1}{r}V'-\zeta^0_l\frac{r_lr_i}{r^3}(-V'+rV'')=0. \tag{34}$$

This equation implies that

$$\zeta^0_i\neq 0,\qquad V=a_2r^2,\qquad \ddot{\zeta}^0_i+2a_2\zeta^0_i=0, \tag{35}$$

$$\dot{\xi}\neq 0,\qquad V=a_2r^2+a_{-2}r^{-2},\qquad \dddot{\xi}+8a_2\dot{\xi}=0, \tag{36}$$

$$\xi^0_{ij}\neq 0,\qquad V=a_2r^2,\qquad \dddot{\xi}^0_{ij}+8a_2\dot{\xi}^0_{ij}=0. \tag{37}$$

In all other cases, ζ^0_i, $\dot{\xi}$ and ξ^0_{ij} are all zero.

From the above we can see that the polynomial symmetry transformation leads to a series of conditions which make it possible to determine the symmetry transformation and the allowed potential simultaneously.

Now we collect and summarise the above results. If the leading-order term of transformation is a bilinear function of r_i, $\dot{r}_i$, then this transformation should be of the form

$$\eta_i=\theta_kr_k\dot{r}_i-\theta_ir_k\dot{r}_k+\rho_k\dot{r}_kr_i-\rho_ir_k\dot{r}_k+\xi\dot{r}_i-\dot{\xi}r_i+\xi^0_{ij}\dot{r}_j-\dot{\xi}^0_{ij}r_j+D_l\varepsilon_{lij}r_j+\zeta^0_i \tag{38}$$

and the function f can be integrated out as

$$f=-\theta_ir_iV+\tfrac{1}{2}\ddot{\xi}r^2+2\xi V+\tfrac{1}{2}\ddot{\xi}^0_{ij}r_ir_j+\xi^0_{ij}r_iV_{,j}-\dot{\zeta}^0_ir_i. \tag{39}$$

Here θ_k and D_l are constant vectors, and ρ_k is arbitrary, but we can ignore it because it does not lead to any constant of the motion. θ_k, ξ, ξ^0_{ij} and ζ^0_i depend on the potentials. There are altogether six cases. We list them as follows.

(1) Arbitrary isotropic potential.

$$V(r)=\text{arbitrary}, \tag{40}$$

$$\xi^0_{ij}=0,\qquad \theta_k=0,\qquad \zeta^0_i=0,\qquad \dot{\xi}=0. \tag{41}$$

(2) Free particle.

$$V(r)=0, \tag{42}$$

$$\theta_i\neq 0,\qquad \dddot{\xi}^0_{ij}=0,\qquad \dddot{\xi}=0,\qquad \ddot{\zeta}^0_i=0. \tag{43}$$

(3) Isotropic harmonic oscillator.

$$V=a_2r^2, \tag{44}$$

$$\theta_i=0,\qquad \dddot{\xi}^0_{ij}+8a_2\dot{\xi}^0_{ij}=0,\qquad \dddot{\xi}+8a_2\dot{\xi}=0,\qquad \ddot{\zeta}^0_i+2a_2\zeta^0_i=0. \tag{45}$$

(4) Coulomb potential.

$$V(r) = a_{-1}/r, \tag{46}$$

$$\theta_i \neq 0, \qquad \zeta_i^0 = 0, \qquad \xi_{ij}^0 = 0, \qquad \dot{\xi} = 0. \tag{47}$$

(5) Centripetal potential.

$$V(r) = a_{-2}/r^2, \tag{48}$$

$$\theta_i = 0, \qquad \xi_{ij}^0 = 0, \qquad \zeta_i^0 = 0, \qquad \dddot{\xi} = 0. \tag{49}$$

(6) Mixed potential.

$$V(r) = a_2 r^2 + a_{-2}/r^2. \tag{50}$$

$$\theta_i = 0, \qquad \xi_{ij}^0 = 0, \qquad \zeta_i^0 = 0, \qquad \dddot{\xi} + 8a_2\dot{\xi} = 0. \tag{51}$$

4. Constants of the motion and symmetry algebras

In this section, we give the explicit transformation solutions for various potentials, present the corresponding constants of the motion and discuss the associated algebras.

(1) Arbitrary isotropic potential.

$$V(r) = \text{arbitrary}, \qquad \eta_i = \xi \dot{r}_i + D_l \varepsilon_{lij} r_j, \qquad f = 2\xi V. \tag{52}$$

ξ and D_l are constants. The transformations are the dilatation and rotation. The corresponding constants of the motion are the energy and the angular momentum:

$$H = \tfrac{1}{2}p^2 + V, \qquad L_i = \varepsilon_{ijk} r_j p_k. \tag{53}$$

This is just a trivial case. We see that the Hamiltonian is a conserved quantity corresponding to the dilatation transformation rather than the time translation (Lévy-Leblond 1971, Mariwalla 1975). The angular momentum operators L_i generate the invariance group SO(3).

(2) Free particle.

$$V = 0,$$

$$\eta_i = \theta_k r_k \dot{r}_i - \theta_i r_k \dot{r}_k + A_{ij}^2(\tfrac{1}{2}t^2\dot{r}_j - tr_j) + A_{ij}^1(t\dot{r}_j - r_j) + A_{ij}^0\dot{r}_j + B_i^1 t + B_i^0 + D_l\varepsilon_{lij}r_j, \tag{54}$$

$$f = A_{ij}^2 r_i r_j - B_i^1 r_i.$$

Here θ_k, A_{ij}^2, A_{ij}^1, A_{ij}^0, B_i^1, B_i^0 and D_l are constants. We obtain the following constants of the motion corresponding to these transformations:

$$\begin{aligned}
\Phi_0^i &= \tfrac{1}{2}(r_i p^2 - p_l r_i p_l - 2Dp_i) = -\tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}(b_j a_l b_m + a_l b_m b_j),\\
\Phi_1 &= \tfrac{1}{2}t^2 H - \tfrac{1}{2}tD + \tfrac{1}{4}r^2 = \tfrac{1}{4}a_l a_l,\\
\Phi_2 &= tH - \tfrac{1}{2}D = \tfrac{1}{4}(a_l b_l + b_l a_l),\\
\Phi_3 &= H = \tfrac{1}{2}p^2 = \tfrac{1}{2}b_l b_l,\\
\Phi_4^{ij} &= t^2(\tfrac{1}{2}p_i p_j - \tfrac{1}{3}\delta_{ij}H) - t(D_{ij} - \tfrac{1}{3}D\delta_{ij}) + \tfrac{1}{2}(r_i r_j - \tfrac{1}{3}r^2\delta_{ij})\\
&= \tfrac{1}{2}a_i a_j - \tfrac{1}{6}a_l a_l \delta_{ij},
\end{aligned} \tag{55}$$

$$\Phi_5^{ij} = t(p_ip_j - \tfrac{2}{3}H\delta_{ij}) - (D_{ij} - \tfrac{1}{3}D\delta_{ij})$$
$$= \tfrac{1}{4}(a_ib_j + a_jb_i + b_ja_i + b_ia_j) - \tfrac{1}{6}(a_lb_l + b_la_l)\delta_{ij},$$
$$\Phi_6^{ij} = p_ip_j - \tfrac{2}{3}H\delta_{ij} = b_ib_j - \tfrac{1}{3}b_lb_l\delta_{ij},$$
$$\Phi_7^i = L_i = \varepsilon_{ilm}r_lp_m = -\varepsilon_{ilm}a_lb_m,$$
$$\Phi_8^i = tp_i - r_i = a_i,$$
$$\Phi_9^i = p_i = b_i.$$

Here, we have divided the tensor constant of the motion into its trace part and its traceless part. The D_{ij} and D are defined as

$$D_{ij} \equiv \tfrac{1}{4}(r_ip_j + r_jp_i + p_jr_i + p_ir_j), \qquad D = D_{ii} = \tfrac{1}{2}(r_ip_i + p_ir_i). \tag{56}$$

From these constants of the motion, it is not difficult to calculate the algebra they close into, using the canonical commutation relation $[r_i, p_j] = \mathrm{i}\delta_{ij}$ or $[a_i, b_j] = -\mathrm{i}\delta_{ij}$. The Φ_0^i, Φ_6^{ij}, Φ_7^i and Φ_9^i all commute with the Hamiltonian. Φ_0^i is the Runge–Lentz vector in this case. The Hamiltonian Φ_3 with Φ_1 and Φ_2 form an SO(2, 1) algebra. Since Φ_3 is not a compact generator, there are no discrete energy levels (Wybourne 1974).

(3) Isotropic harmonic oscillator. (Take positive a_2 as an example.)

$$V(r) = a_2r^2 = \tfrac{1}{2}\omega^2r^2, \tag{57}$$

$$\begin{aligned}\eta_i &= A_{ij}[\cos(2\omega t)\dot{r}_j + 2\omega\sin(2\omega t)r_j] + A'_{ij}[\sin(2\omega t)\dot{r}_j - 2\omega\cos(2\omega t)r_j] \\ &\quad + A^0_{ij}\dot{r}_j + D_l\varepsilon_{lij}r_j + C_i^1\cos\omega t + C_i^2\sin\omega t,\end{aligned} \tag{58}$$

$$\begin{aligned}f &= A_{ij}(-\omega^2)\cos(2\omega t)r_ir_j + A'_{ij}(-\omega^2)\sin(2\omega t)r_ir_j + A^0_{ij}\omega^2r_ir_j \\ &\quad + C_i^1\omega\sin(\omega t)r_i - C_i^2\omega\cos(\omega t)r_i.\end{aligned}$$

The corresponding constants of the motion are

$$\begin{aligned}
\Phi_1 &= (H - \omega^2r^2)\cos 2\omega t + \omega D\sin 2\omega t = \tfrac{1}{2}(c_{1i}c_{1i} - c_{2i}c_{2i}),\\
\Phi_2 &= (H - \omega^2r^2)\sin 2\omega t - \omega D\cos 2\omega t = \tfrac{1}{2}(c_{1i}c_{2i} + c_{2i}c_{1i}),\\
\Phi_3 &= H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2r^2 = \tfrac{1}{2}(c_{1i}c_{1i} + c_{2i}c_{2i}),\\
\Phi_4^{ij} &= [\tfrac{1}{2}p_ip_j - \tfrac{1}{2}\omega^2r_ir_j - \tfrac{1}{3}(H - \omega^2r^2)\delta_{ij}]\cos 2\omega t + \omega(D_{ij} - \tfrac{1}{3}D\delta_{ij})\sin 2\omega t\\
&= \tfrac{1}{2}(c_{1i}c_{1j} - c_{2i}c_{2j}) - \tfrac{1}{6}(c_{1l}c_{1l} - c_{2l}c_{2l})\delta_{ij},\\
\Phi_5^{ij} &= [\tfrac{1}{2}p_ip_j - \tfrac{1}{2}\omega^2r_ir_j - \tfrac{1}{3}(H - \omega^2r^2)\delta_{ij}]\sin 2\omega t - \omega(D_{ij} - \tfrac{1}{3}D\delta_{ij})\cos 2\omega t\\
&= \tfrac{1}{4}(c_{1i}c_{2j} + c_{1j}c_{2i} + c_{2i}c_{1j} + c_{2j}c_{1i}) - \tfrac{1}{6}(c_{1l}c_{2l} + c_{2l}c_{1l})\delta_{ij},\\
\Phi_6^{ij} &= \tfrac{1}{2}p_ip_j + \tfrac{1}{2}\omega^2r_ir_j - \tfrac{1}{3}H\delta_{ij} = \tfrac{1}{2}(c_{1i}c_{1j} + c_{2i}c_{2j}) - \tfrac{1}{6}(c_{1l}c_{1l} + c_{2l}c_{2l})\delta_{ij},\\
\Phi_7^i &= L_i = \varepsilon_{ijk}r_jp_k = (1/\omega)\varepsilon_{ijk}c_{1j}c_{2k},\\
\Phi_8^i &= p_i\cos\omega t + \omega\sin(\omega t)r_i = c_{1i},\\
\Phi_9^i &= p_i\sin\omega t - \omega\cos(\omega t)r_i = c_{2i}.
\end{aligned} \tag{59}$$

To display the SU(3) symmetry, we define the following appropriate linear combinations:

$$\begin{aligned}c_i &= (2\omega)^{-1/2}(c_{1i} + \mathrm{i}c_{2i}) = (2\omega)^{-1/2}\,\mathrm{e}^{\mathrm{i}\omega t}(p_i - \mathrm{i}\omega r_i),\\ c_i^\dagger &= (2\omega)^{-1/2}(c_{1i} - \mathrm{i}c_{2i}) = (2\omega)^{-1/2}\,\mathrm{e}^{-\mathrm{i}\omega t}(p_i + \mathrm{i}\omega r_i).\end{aligned} \tag{60}$$

Then the canonical quantisation condition leads to

$$[c_i, c_j^\dagger] = \delta_{ij}. \tag{61}$$

In terms of c_i, $c_i^\dagger$, the above constraints of motion can be rewritten in the form

$$\begin{aligned}
&\Phi_1 = \tfrac{1}{2}\omega(c_ic_i + c_i^\dagger c_i^\dagger),\\
&\Phi_2 = -\tfrac{1}{2}\mathrm{i}\omega(c_ic_i - c_i^\dagger c_i^\dagger),\\
&\Phi_3 = \tfrac{1}{2}\omega(c_ic_i^\dagger + c_i^\dagger c_i) = \omega(c_i^\dagger c_i + \tfrac{3}{2}),\\
&\Phi_4^{ij} = \tfrac{1}{2}\omega[c_ic_j + c_i^\dagger c_j^\dagger - \tfrac{1}{3}(c_lc_l + c_l^\dagger c_l^\dagger)\delta_{ij}],\\
&\Phi_5^{ij} = -\mathrm{i}\tfrac{1}{2}\omega[c_ic_j - c_i^\dagger c_j^\dagger - \tfrac{1}{3}(c_lc_l - c_l^\dagger c_l^\dagger)\delta_{ij}],\\
&\Phi_6^{ij} = \tfrac{1}{4}\omega[c_ic_j^\dagger + c_i^\dagger c_j + c_jc_i^\dagger + c_j^\dagger c_i - \tfrac{2}{3}(c_lc_l^\dagger + c_l^\dagger c_l)\delta_{ij}],\\
&\Phi_7^i = \mathrm{i}\varepsilon_{ijk}c_jc_k^\dagger,\\
&\Phi_8^i = (\omega/2)^{1/2}(c_i + c_i^\dagger),\\
&\Phi_9^i = -\mathrm{i}(\omega/2)^{1/2}(c_i - c_i^\dagger).
\end{aligned} \tag{62}$$

It is obvious that the c_i and $c_i^\dagger$ are the annihilation and creation operators respectively. The symmetric part Φ_6^{ij} and the antisymmetric part Φ_7^i of a traceless tensor operator which is commutative with the Hamiltonian form the invariance group of SU(3) (Wybourne 1974), while the Φ_1, Φ_2 and the Hamiltonian close in SO(2, 1) with H being the compact generator. In contrast to the conventional situation (Baker 1956, Goshen and Lipkin 1959, Katzin 1973, Wybourne 1974), here the creation and annihilation operators are also the conserved quantities.

(4) Coulomb potential.

$$V = a_{-1}/r, \tag{63}$$

$$\begin{aligned}
&\eta_i = \theta_k r_k \dot{r}_i - \theta_i r_k \dot{r}_k + \xi \dot{r}_i + D_l \varepsilon_{lij} r_j,\\
&f = -\theta_i r_i a_{-1}/r + 2\xi a_{-1}/r.
\end{aligned} \tag{64}$$

The transformation characterised by a vector parameter θ_i is $\delta \boldsymbol{r} = \boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\theta})$; $\boldsymbol{r}$ changes in the direction perpendicular to itself, so it is a sort of rotating transformation. Corresponding to (64), we have the following constants of the motion:

$$\begin{aligned}
&\Phi_1^i = Dp_i - \tfrac{1}{2}(r_ip^2 + p_lr_ip_l) + a_{-1}r_i/r,\\
&\Phi_2^i = L_i = \varepsilon_{ijk}r_jp_k,\\
&\Phi_3 = H = \tfrac{1}{2}p^2 + a_{-1}/r.
\end{aligned} \tag{65}$$

Obviously, Φ_1^i is nothing but the Runge–Lentz vector (Runge 1919, Lentz 1924) which, together with the angular momentum vector, generates the invariance group SO(4) of the hydrogen atom (Fock 1935, Bargman 1936). Unlike some authors (Katzin 1973, Fradkin 1967, Prince and Eliezer 1981), we obtain this vector from Noether's theorem directly.

(5) Centripetal potential (Jackiw 1972).

$$V = a_{-2}/r^2, \tag{66}$$

$$\begin{aligned}
&\eta_i = A_2(\tfrac{1}{2}t^2\dot{r}_i - tr_i) + A_1(t\dot{r}_i - r_i) + A_0\dot{r}_i,\\
&f = \tfrac{1}{2}A_2r^2 + (A_2t^2 + 2A_1t + 2A_0)a_{-2}/r^2.
\end{aligned} \tag{67}$$

We obtain the following constants of the motion:

$$\Phi_1 = t^2H - \tfrac{1}{2}tD + \tfrac{1}{4}r^2, \qquad \Phi_2 = tH - \tfrac{1}{4}D,$$
$$\Phi_3 = H = \tfrac{1}{2}p^2 + a_{-2}/r^2, \qquad \Phi_4^i = L_i = \varepsilon_{ijk}r_jp_k. \tag{68}$$

The invariance group is the geometrical rotation group SO(3). Φ_1, Φ_2 and Φ_3 form a non-invariance group SO(2, 1), but the Hamiltonian is not the compact generator. Therefore, the symmetry group of the centripetal potential is SO(2, 1)×SO(3) ~ SO(3, 1).

(6) Mixed potential.

$$V(r) = a_2r^2 + a_{-2}/r^2 = \tfrac{1}{2}\omega^2r^2 + a_{-2}/r^2, \tag{69}$$

$$\eta_i = A_1(\dot{r}_i \cos 2\omega t + 2\omega r_i \sin 2\omega t) + A_2(\dot{r}_i \sin 2\omega t - 2\omega r_i \cos 2\omega t) + A_0\dot{r}_i + D_l\varepsilon_{lij}r_j,$$
$$f = A_1(-\omega^2r^2 + 2a_{-2}/r^2)\cos 2\omega t + A_2(-\omega^2r^2 + a_{-2}/r^2)\sin 2\omega t + A_0(\omega r^2 + 2a_{-2}/r^2). \tag{70}$$

Following these transformations, we obtain the conserved quantities:

$$\Phi_1 = (H - \omega^2r^2)\cos 2\omega t + \omega D \sin 2\omega t, \qquad \Phi_2 = (H - \omega^2r^2)\sin 2\omega t - \omega D \cos 2\omega t,$$
$$\Phi_3 = H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2r^2 + a_{-2}/r^2, \qquad \Phi_4^i = L_i = \varepsilon_{ijk}r_jp_k. \tag{71}$$

The symmetry is the same as in case (5), but here the Hamiltonian is the compact generator of SO(2, 1) (Zhu Dongpei 1981). We see that, due to the existence of the centripetal part, the symmetry of the harmonic oscillator has been reduced.

5. Remarks

In the above we have shown that Noether's theorem provides a systematic way to solve the dynamical symmetry problem. In terms of the polynomial symmetry transformation we can determine all possible allowed potentials and transformations, and find out the constants of the motion and the corresponding symmetry algebra, hence solving the dynamical symmetry problem.

If we restrict ourselves to point transformations, the possible isotropic potentials which possess symmetry larger than geometrical symmetry are the free particle, the harmonic oscillator, the centripetal potential and the mixed potential. When we go one step further, allowing the transformation to contain the velocity linearly, the Coulomb potential enters the scene. If we increase the power of the transformation polynomial, can we find more potentials possessing dynamical symmetry? Or, conversely, can we prove a theorem like Bertrand's theorem in classical mechanics (Greenberg 1966) to the effect that the harmonic oscillator and the Coulomb potential are the only systems which admit the invariance symmetry larger than SO(3)?

This problem can be stated in another way. For the free particle and the oscillator, there are fundamental constants of the motion (a_i, b_i, c_{1i} and c_{2i}) from which any high-order conserved quantities (including the Hamiltonian) can be constructed. This is, in a sense, the converse to the related-integral theorem (Katzin and Levine 1968, Katzin 1973). So, essentially, we do not need any polynomial transformations higher than point transformations for these systems. In other cases, the known fundamental constants of the motion are the Hamiltonian and the angular momentum. So, searching for dynamical symmetry is equivalent to seeking the additional constants of the motion which are not simply the polynomial combinations of the Hamiltonian and the angular momentum. In the Coulomb potential and the centripetal potential cases we have seen such examples of this. Are there other cases? This is an interesting problem, but the calculations may be very complicated.

We have seen how Noether's theorem is extremely useful for attacking the dynamical symmetry problem. Beyond the isotropic system, other systems are also studied, such as non-stationary systems, damping systems, multi-particle systems, and so on. Recently, Noether's theorem has been used to obtain the constants of the motion for a time-dependent oscillator (Lutzky 1978a, b, Ray and Reid 1979, Prince and Eliezer 1980, Leach 1981). Thus, investigation of these types of systems through the techniques discussed appears worthwhile.

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